DEFORMATION OF MICROSCOPICALLY NONHOMOGENEOUS ELASTIC BODIES

(O DEFORMIROVANII MIKRONEODNORODNYKH UPRUGIKH TEL)

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The problem studied is that of the determination of the statistical displacement field characteristics of the stress and strain in an anisotropic microscopically nonhomogeneous elastic body in a macroscopically homogeneous state of strain. The initial statistically nonlinear boundary value problem is linearized by the small parameter method, and a solution in terms of the statistical characteristics of the field of elastic moduli is given. The case of statistical isotropy of this field is considered.

1. We consider a solid strained anisotropic, microscopically nonhomogeneous body (e.g. a polycrystalline body) in which the microscopic nonhomogeneity has a random character. Hooke's law is written in the form

$$\tau_{ij} = c_{ijlm} e_{lm} \tag{1.1}$$

Here τ_{ij} is the stress tensor, e_{lm} is the tensor for small strains, and c_{ijlm} is the tensor defining the elastic properties of the medium. For the considered microscopically nonhomogeneous body, the components of the tensor c_{ijlm} are random functions of the x_s coordinates and the tensor itself determine the random tensor field, statistical description of which is analogous to the description of a tensor of the second rank [1].

Along with the mean value $\langle c_{ijlm}\rangle$ of the tensor c_{ijlm} the moment of interaction of the values of the tensor field at two points plays a most important role

$$c_{ijlm}^{prst}(x_s^1, x_s^2) = \langle c_{ijlm}(x_s^1)\, c_{prst}(x_s^2)\rangle, \qquad c_{ijlm} = c_{ijlm} - \langle c_{ijlm}\rangle \tag{1.2}$$

Here and below the angle brackets denote the statistical mean of the corresponding quantities. By virtue of the known symmetry of the tensor c_{ijlm}, the following conditions hold for the interaction moment (1.2) :

$$c_{ijlm}^{prst} = c_{jilm}^{prst} = c_{jiml}^{prst} = c_{lmij}^{prst} = c_{lmij}^{rpst} = c_{lmij}^{rpts} = c_{lmij}^{stpr} \tag{1.3}$$

For the case of statistically homogeneous field to which we will limit our considerations, the mean values $\langle c_{ijlm}\rangle$ of the field are constant and the interaction moment (1.2), also called the correlation tensor, will be a function of a single vector ξ,

$$c_{ijlm}^{prst} = c_{ijlm}^{prst}(\xi_s), \qquad \xi_s = x_s^2 - x_s^1 \tag{1.4}$$

in which the relation

$$c_{ijlm}^{prst}(\xi_s) = c_{prst}^{ijlm}(-\xi_s) \tag{1.5}$$

holds.

The strains e_{lm} are connected with the disclacements w_l by the relations

$$e_{lm} = \frac{1}{2}\left(\frac{\partial w_l}{\partial x_m} + \frac{\partial w_m}{\partial x_l}\right) \tag{1.6}$$

We introduce the notation

$$u_l = \langle w_l \rangle, \quad v_l = w_l - u_l, \quad \varepsilon_{lm} = \langle e_{lm} \rangle, \quad \gamma_{lm} = e_{lm} - \varepsilon_{lm}$$
$$\sigma_{ij} = \langle \tau_{ij} \rangle, \qquad p_{ij} = \tau_{ij} - \sigma_{ij} \tag{1.7}$$

Then, along with (1.6) we have

$$\varepsilon_{lm} = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\right), \ \gamma_{lm} = \frac{1}{2}\left(\frac{\partial v_l}{\partial x_m} + \frac{\partial v_m}{\partial x_l}\right) \tag{1.8}$$

We consider a body of volume v bounded by a surface s, in a state of strain such that $\varepsilon_{lm} = \text{const}$. On the assumption of ergodicity of the random functions c_{ijlm}, this means that the body is in a macroscopically homogeneous state of strain. Then

$$u_l = \varepsilon_{lm} x_m$$

For the body the dimensions of which are very large in comparison with the scale of nonhomogeneities of the tensor c_{ijlm}, $v_l \ll u_l$ on the boundary of the body; therefore the boundary condition may be written in the form

$$w_l|_s = u_l|_s = \varepsilon_{lm} x_m|_s \tag{1.9}$$

We get the boundary value problem for determination of w_l by adding the equation of equilibrium (in the absence of body forces)

$$\partial \tau_{ij} / \partial x_j = 0 \tag{1.10}$$

to Equations (1.1), (1.6) and (1.9).

We suppose further that the field c_{ijlm} may be représented in the form

$$c_{ijlm} = \langle c_{ijlm} \rangle + \alpha b_{ijlm} \qquad (\alpha b_{ijlm} = c_{ijlm}) \tag{1.11}$$

where the b_{ijlm} are random restricted functions of the coordinates and α is a small parameter, not of a random character. Then, after taking account of (1.7) and (1.8) and of the symmetry of the tensor c_{ijlm}, (1.1) may be presented in the form

$$\tau_{ij} = (\langle c_{ijlm} \rangle + \alpha b_{ijlm})\left(\varepsilon_{lm} + \frac{\partial v_l}{\partial x_m}\right) \tag{1.12}$$

From (1.10), (1.12) and (1.7), (1.9) we obtain the boundary value problem for the determination of the vector v_l

$$\langle c_{ijlm} \rangle \frac{\partial^2 v_l}{\partial x_j \partial x_m} = -\alpha \frac{\partial}{\partial x_j}\left[b_{ijlm}\left(\varepsilon_{lm} + \frac{\partial v_l}{\partial x_m}\right)\right], \qquad v_l|_s = 0 \tag{1.13}$$

Here the macroscopic strains ε_{lm} are considered as given.

2. By virtue of the randomness of the tensor b_{ijlm} and the vector v_l the boundary value problem (1.13) is statistically nonlinear. It is linearized if its solution is represented in the form of a series in powers of the small parameter α

$$v_l = \sum_{k=0}^{\infty} \alpha^k v_l^{(k)} \tag{2.1}$$

By substitution of (2.1) into (1.13) and equating coefficients of the equal powers of α, we find

$$\langle c_{ijlm}\rangle \frac{\partial^2 v_l^{(0)}}{\partial x_j \partial x_m} = 0, \qquad v_l^{(0)}|_s = 0 \tag{2.2}$$

$$\langle c_{ijlm}\rangle \frac{\partial^2 v_l^{(1)}}{\partial x_j \partial x_m} = -\frac{\partial}{\partial x_j}\left[b_{ijlm}\left(\varepsilon_{lm} + \frac{\partial v_l^{(0)}}{\partial x_m}\right)\right], \qquad v_l^{(1)}|_s = 0 \tag{2.3}$$

$$\langle c_{ijlm}\rangle \frac{\partial^2 v_l^{(k)}}{\partial x_j \partial x_m} = -\frac{\partial}{\partial x_j}\left(b_{ijlm} \frac{\partial v_l^{(k-1)}}{\partial x_m}\right), \qquad v_l^{(k)}|_s = 0 \qquad (k = 2, 3, \ldots)$$

Because of the uniqueness of the boundary value problem (2.2) we have $v_l^{(0)} = 0$, and (2.3) is finally written in the form

$$\langle c_{ijlm}\rangle \frac{\partial^2 v_l^{(1)}}{\partial x_j \partial x_m} = -\varepsilon_{lm} \frac{\partial b_{ijlm}}{\partial x_j}, \qquad v_l^{(1)}|_s = 0$$

$$\langle c_{ijlm}\rangle \frac{\partial^2 v_l^{(k)}}{\partial x_j \partial x_m} = -\frac{\partial}{\partial x_j}\left(b_{ijlm} \frac{\partial v_l^{(k-1)}}{\partial x_m}\right), \qquad v_l^{(k)}|_s = 0 \qquad (k = 2, 3, \ldots) \tag{2.4}$$

Relations (2.4) represent in themselves successive recurrent statistically linear boundary value problems determining the terms in the expansion (2.1).

By representation of the solutions to the boundary value problems (2.4) in terms of a Green tensor $G_{ip}(x_s, x_s^1)$, which is one and the same for all the problems, we have [2]

$$v_i^{(1)}(x_s) = \varepsilon_{lm} \int_{(v)} G_{in}(x_s, x_s^1) \frac{\partial b_{njlm}(x_s^1)}{\partial x_j^1} dv_1$$

$$v_i^{(k)}(x_s) = \int_{(v)} G_{in}(x_s, x_s^1) \frac{\partial}{\partial x_j^1}\left[b_{njlm}(x_s^1) \frac{\partial v_l^{(k-1)}(x_s^1)}{\partial x_m^1}\right] dv_1 \qquad (k = 2, 3, \ldots) \tag{2.5}$$

The functions $v_i^{(k)}$ $(k = 2, 3, \ldots)$ may be expressed in terms of the tensors G_{in} and b_{njem}. For example, for $v_i^{(2)}$, we find

$$v_i^{(2)}(x_s) = \varepsilon_{st} \int_{(v)}\int_{(v)} G_{in}(x_s, x_s^1) \frac{\partial G_{lp}(x_s^1, x_s^2)}{\partial x_m^1} \frac{\partial^2}{\partial x_j^1 \partial x_r^2} [b_{njlm}(x_s^1) b_{prst}(x_s^2)] dv_1 dv_2 +$$

$$+ \varepsilon_{st} \int_{(v)}\int_{(v)} G_{in}(x_s, x_s^1) \frac{\partial^2 G_{lp}(x_s^1, x_s^2)}{\partial x_j^1 \partial x_m^1} \frac{\partial}{\partial x_r^2} [b_{njlm}(x_s^1) b_{prst}(x_s^2)] dv_1 dv_2$$

It is seen from the structure of Formulas (2.5) that the functions $v_i^{(k)}$ for any k are linear functions of the mean strains ε_{lm}. We find therefore for v_i (2.1)

$$v_i(x_s) = \varepsilon_{st} \sum_{k=1}^{\infty} \varphi_{ist}^{(k)}(x_s) \tag{2.6}$$

The quantities $\varphi_{ist}^{(k)}(x_s)$ are determined by the Green tensor G_{in} and the deviation tensor c'_{ijlm}. for the elastic moduli by virtue of (2.5) and (1.11). In particular, we have

$$\varphi_{ist}^{(1)}(x_s) = \int\limits_{(v)} G_{in}(x_s, x_s^1) \frac{\partial c'_{njst}(x_s^1)}{\partial x_j^1} dv_1$$

$$\varphi_{ist}^{(2)}(x_s) = \int\limits_{(v)}\int\limits_{(v)} G_{in}(x_s, x_s^1) \frac{\partial G_{lp}(x_s^1, x_s^2)}{\partial x_m^1} \frac{\partial^2}{\partial x_j^1 \partial x_r^2} [c'_{njlm}(x_s^1) c'_{prst}(x_s^2)]\, dv_1 dv_2 +$$

$$+ \int\limits_{(v)}\int\limits_{(v)} G_{in}(x_s, x_s^1) \frac{\partial^2 G_{lp}(x_s^1, x_s^2)}{\partial x_j^1 \partial x_m^1} \frac{\partial}{\partial x_r^2} [c'_{njlm}(x_s^1) c'_{prst}(x_s^2)]\, dv_1 dv_2$$

Having the solution of (2.6) it is easy to find the statistical characteristics of the vector displacements. In particular, for the moments of displacements of the nth order

$$v_{i_1 \cdots i_n} = \langle v_{i_1}(x_s^1) \cdots v_{i_n}(x_s^n) \rangle$$

we have

$$v_{i_1 \cdots i_n} = \varepsilon_{s_1 t_1} \cdots \varepsilon_{s_n t_n} \sum_{k_1, \ldots, k_n = 1}^{\infty} \langle \varphi_{i_1 s_1 t_1}^{(k_1)}(x_s^1) \cdots \varphi_{i_n s_n t_n}^{(k_n)}(x_s^n) \rangle$$

From this we find the second order moments for $n = 2$

$$v_{ij}(x_s^1, x_s^2) = \varepsilon_{pr} \varepsilon_{st} \sum_{k, l=1}^{\infty} \langle \varphi_{ipr}^{(k)}(x_s^1) \varphi_{jst}^{(e)}(x_s^2) \rangle \tag{2.7}$$

3. We find the statistical characteristics of the stress tensor [1]

$$\sigma_{ij} = \langle \tau_{ij} \rangle, \qquad p_{i_1 j_1 \cdots i_n j_n} = \langle p_{i_1 j_1}(x_s^1) \cdots p_{i_n j_n}(x_s^n) \rangle \qquad (p_{ij} = \tau_{ij} - \sigma_{ij}) \tag{3.1}$$

We proceed from the relations (1.12), which we rewrite in the form

$$\tau_{ij} = \langle c_{ijlm} \rangle \varepsilon_{lm} + c'_{ijlm} \varepsilon_{lm} + \langle c_{ijlm} \rangle \frac{\partial v_l}{\partial x_m} + c_{ijlm} \frac{\partial v_l}{\partial x_m} \tag{3.2}$$

We find from (3.2)

$$\sigma_{ij} = \langle c_{ijlm} \rangle \varepsilon_{lm} + \left\langle c'_{ijlm} \frac{\partial v_l}{\partial x_m} \right\rangle$$

$$p_{ij} = c'_{ijlm} \varepsilon_{lm} + \langle c_{ijlm} \rangle \frac{\partial v_l}{\partial x_m} + c'_{ijlm} \frac{\partial v_l}{\partial x_m} - \left\langle c'_{ijlm} \frac{\partial v_l}{\partial x_m} \right\rangle \tag{3.3}$$

We have from (2.6)

$$\frac{\partial v_l}{\partial x_m} = \varepsilon_{st} \varphi_{lstm}, \qquad \varphi_{lstm} = \sum_{k=1}^{\infty} \frac{\partial \varphi_{lst}^{(k)}(x_s)}{\partial x_m} \tag{3.4}$$

By introduction of the notation

$$\psi_{ijst} = c'_{ijlm} \varphi_{lstm} \tag{3.5}$$

we obtain from (3.3)

$$\sigma_{ij} = (\langle c_{ijst}\rangle + \langle\psi_{ijst}\rangle)\,\varepsilon_{st}, \qquad p_{ij} = \eta_{ijst}\varepsilon_{st}$$
$$\eta_{ijst} = c_{ijst} + \langle c_{ijlm}\rangle\,\varphi_{lstm} + \psi_{ijst} - \langle\psi_{ijst}\rangle \tag{3.6}$$

The nth order moment of the stress tensor (3.1) is written in the form

$$p_{i_1j_1\cdots i_nj_n} = \langle\eta_{i_1j_1s_1t_1}(x_s^1)\cdots\eta_{i_nj_ns_nt_n}(x_s^n)\rangle\,\varepsilon_{s_1t_1}\cdots\varepsilon_{s_nt_n}$$

In particular, for $n = 2$ we have

$$p_{ijkl}(x_s^1, x_s^2) = \langle\eta_{ijpr}(x_s^1)\,\eta_{klst}(x_s^2)\rangle\,\varepsilon_{pr}\varepsilon_{st} \tag{3.7}$$

For certain purposes it is necessary to establish the connection between the 2nd order moments of stress and strain

$$p_{ijkl} = \langle p_{ij}(x_s^1)\,p_{kl}(x_s^2)\rangle, \qquad \gamma_{ijkl} = \langle\gamma_{ij}(x_s^1)\,\gamma_{kl}(x_s^2)\rangle$$

We get, after transformations and after making use of (3.3) to (3.5)

$$p_{ijlm} = \langle c_{ijpr}\rangle\langle c_{lmst}\rangle\,\gamma_{prst} + (\mu_{ijpr}^{lmst} + \nu_{ijpr}^{lmst})\,\varepsilon_{pr}\varepsilon_{st} \tag{3.8}$$

Here

$$\mu_{ijpr}^{lmst} = c_{ijpr}^{lmst} + \langle c_{ijkn}\rangle\langle c_{lmst}^{\prime}(x_s^2)\,\varphi_{kprn}(x_s^1)\rangle + \langle c_{lmkn}\rangle\langle c_{ijpr}^{\prime}(x_s^1)\,\varphi_{kstn}(x_s^2)\rangle$$
$$\nu_{ijpr}^{lmst} = \langle c_{ijpr}^{\prime}(x_s^1)\,\psi_{lmst}(x_s^2)\rangle + \langle c_{lmst}^{\prime}(x_s^2)\,\psi_{ijpr}(x_s^1)\rangle +$$
$$+ \langle c_{ijkn}\rangle\langle\varphi_{kprn}(x_s^1)\,\psi_{lmst}(x_s^2)\rangle + \langle c_{lmkn}\rangle\langle\varphi_{kstn}(x_s^2)\,\psi_{ijpr}(x_s^1)\rangle +$$
$$+ \langle\psi_{ijpr}(x_s^1)\,\psi_{lmst}(x_s^2)\rangle - \langle\psi_{ijpr}(x_s^1)\rangle\langle\psi_{lmst}(x_s^2)\rangle \tag{3.9}$$

The tensors μ_{ijpr}^{lmst} and ν_{ijpr}^{lmst} in (3.9) are determined by the Green tensors of the original problem and the statistical properties of the elastic moduli fields c_{ijlm}, in which if the expansion (3.4) is limited to only the first term, the values of μ depend only on the second order moments of the tensor c_{ijlm} and the values of ν on the third or fourth order moments. If moments of higher order may be neglected, i.e. if the condition

$$|\nu_{ijpr}^{lmst}| \ll |\mu_{ijpr}^{lmst}| \tag{3.10}$$

holds, then relation (3.8) takes the form

$$p_{ijlm} = \langle c_{ijpr}\rangle\langle c_{lmst}\rangle\,\gamma_{prst} + \mu_{ijpr}^{lmst}\varepsilon_{pr}\varepsilon_{st} \tag{3.11}$$

Condition (3.10) is satisfied, in particular, in case of small microscopic nonhomogeneity when the deviations c'_{ijkl} of the elastic moduli are small compared with their mean value $\langle c_{ijkl}\rangle$, i.e. if

$$|c_{ijkl}^{\prime}| \ll |\langle c_{ijkl}\rangle|$$

4. We now assume the body under consideration to be unbounded, and the field c_{ijlm} of elastic moduli to be statistically isotropic [1]. In this case the tensor $\langle c_{ijlm}\rangle$ will be the isotropic tensor

$$\langle c_{ijkl}\rangle = \mu_1\delta_{ij}\delta_{kl} + \mu_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where δ_{ik} is a unit tensor of the second rank and the Green tensor G_{km} in (2.5) may be written in explicit 'orm [2]

$$G_{km} = \frac{1}{8\pi\mu_2}\left[\frac{2}{r}\delta_{km} - \frac{\mu_1+\mu_2}{\mu_1+2\mu_2}\frac{\partial^2 r}{\partial x_k\,\partial x_m}\right], \qquad r^2 = (x_j - x_j^1)(x_j - x_j^1) \tag{4.1}$$

By using the method expounded by Robertson [3], we find further that for a statistically isotropic field c_{ijkl} under conditions (1.3) and (1.5), the correlation tensor (1.4) has the form

$$
\begin{aligned}
c_{ijkl}^{prst}(\xi_s) = {} & c_1(\rho)\,\xi_i\xi_j\xi_k\xi_l\xi_p\xi_r\xi_s\xi_t + \qquad (4.2)\\
& + c_2(\rho)(\delta_{ij}\xi_k\xi_l\xi_p\xi_r\xi_s\xi_t + \delta_{kl}\xi_i\xi_j\xi_p\xi_r\xi_s\xi_t + \delta_{pr}\xi_s\xi_l\xi_i\xi_j\xi_k\xi_l + \delta_{st}\xi_p\xi_r\xi_i\xi_j\xi_k\xi_l) + \\
& + c_3(\rho)(\delta_{ik}\xi_j\xi_l\xi_p\xi_r\xi_s\xi_t + \delta_{il}\xi_j\xi_k\xi_p\xi_r\xi_s\xi_t + \delta_{jk}\xi_i\xi_l\xi_p\xi_r\xi_s\xi_t + \\
& + \delta_{jl}\xi_i\xi_k\xi_p\xi_r\xi_s\xi_t + \delta_{ps}\xi_r\xi_t\xi_i\xi_j\xi_k\xi_l + \delta_{pt}\xi_r\xi_s\xi_i\xi_j\xi_k\xi_l + \delta_{rs}\xi_p\xi_t\xi_i\xi_j\xi_k\xi_l + \\
& + \delta_{rt}\xi_p\xi_s\xi_i\xi_j\xi_k\xi_l) + c_4(\rho)(\delta_{ij}\delta_{kl}\xi_p\xi_r\xi_s\xi_t + \delta_{pr}\delta_{st}\xi_i\xi_j\xi_k\xi_l) + \\
& + c_5(\rho)(\delta_{ik}\delta_{jl}\xi_p\xi_r\xi_s\xi_t + \delta_{il}\delta_{jk}\xi_p\xi_r\xi_s\xi_t + \delta_{ps}\delta_{rt}\xi_i\xi_j\xi_k\xi_l + \\
& + \delta_{pt}\delta_{rs}\xi_i\xi_j\xi_k\xi_l) + c_6(\rho)(\delta_{ij}\delta_{pr}\xi_k\xi_l\xi_s\xi_t + \delta_{ij}\delta_{st}\xi_k\xi_l\xi_p\xi_r + \\
& + \delta_{kl}\delta_{pr}\xi_i\xi_j\xi_s\xi_t + \delta_{kl}\delta_{st}\xi_i\xi_j\xi_p\xi_r) + c_7(\rho)(\delta_{ij}\delta_{ps}\xi_k\xi_l\xi_r\xi_t + \delta_{ij}\delta_{pt}\xi_k\xi_l\xi_r\xi_s + \\
& + \delta_{ij}\delta_{rs}\xi_k\xi_l\xi_p\xi_t + \delta_{ij}\delta_{rt}\xi_k\xi_l\xi_p\xi_s + \delta_{ik}\delta_{pr}\xi_j\xi_l\xi_s\xi_t + \delta_{ik}\delta_{st}\xi_j\xi_l\xi_p\xi_r + \\
& + \delta_{il}\delta_{pr}\xi_j\xi_k\xi_s\xi_t + \delta_{il}\delta_{st}\xi_j\xi_k\xi_p\xi_r + \delta_{jk}\delta_{pr}\xi_i\xi_l\xi_s\xi_t + \delta_{jk}\delta_{sl}\xi_i\xi_l\xi_p\xi_r + \\
& + \delta_{jl}\delta_{pr}\xi_i\xi_k\xi_s\xi_t + \delta_{jl}\delta_{st}\xi_i\xi_k\xi_p\xi_r + \delta_{kl}\delta_{ps}\xi_i\xi_j\xi_r\xi_t + \delta_{kl}\delta_{pt}\xi_i\xi_j\xi_r\xi_s + \\
& + \delta_{kl}\delta_{rs}\xi_i\xi_j\xi_p\xi_t + \delta_{kl}\delta_{rt}\xi_i\xi_j\xi_p\xi_s) + c_8(\rho)(\delta_{ik}\delta_{ps}\xi_j\xi_l\xi_r\xi_t + \\
& + \delta_{ik}\delta_{pt}\xi_j\xi_l\xi_r\xi_s + \delta_{ik}\delta_{rs}\xi_j\xi_l\xi_p\xi_t + \delta_{ik}\delta_{rt}\xi_j\xi_l\xi_p\xi_s + \delta_{il}\delta_{ps}\xi_j\xi_k\xi_r\xi_t + \\
& + \delta_{il}\delta_{pt}\xi_j\xi_k\xi_r\xi_s + \delta_{il}\delta_{rs}\xi_j\xi_k\xi_p\xi_t + \delta_{il}\delta_{rt}\xi_j\xi_k\xi_p\xi_s + \delta_{jk}\delta_{ps}\xi_i\xi_l\xi_r\xi_t + \\
& + \delta_{jk}\delta_{pt}\xi_i\xi_l\xi_r\xi_s + \delta_{jk}\delta_{rs}\xi_i\xi_l\xi_p\xi_t + \delta_{jk}\delta_{rt}\xi_i\xi_l\xi_p\xi_s + \delta_{jl}\delta_{ps}\xi_i\xi_k\xi_r\xi_t + \\
& + \delta_{jl}\delta_{pt}\xi_i\xi_k\xi_r\xi_s + \delta_{jl}\delta_{rs}\xi_i\xi_k\xi_p\xi_t + \delta_{jl}\delta_{rt}\xi_i\xi_k\xi_p\xi_s) + c_9(\rho)(\delta_{ij}\delta_{kl}\delta_{pr}\xi_s\xi_t + \\
& + \delta_{ij}\delta_{kl}\delta_{st}\xi_p\xi_r + \delta_{pr}\delta_{st}\delta_{ij}\xi_k\xi_l + \delta_{pr}\delta_{st}\delta_{kl}\xi_i\xi_j) + c_{10}(\rho)(\delta_{ij}\delta_{kl}\delta_{ps}\xi_r\xi_t + \\
& + \delta_{ij}\delta_{kl}\delta_{pt}\xi_r\xi_s + \delta_{ij}\delta_{kl}\delta_{rs}\xi_p\xi_t + \delta_{ij}\delta_{kl}\delta_{rt}\xi_p\xi_s + \delta_{pr}\delta_{st}\delta_{ik}\xi_j\xi_l + \\
& + \delta_{pr}\delta_{st}\delta_{il}\xi_j\xi_k + \delta_{pr}\delta_{st}\delta_{jk}\xi_i\xi_l + \delta_{pr}\delta_{st}\delta_{jl}\xi_i\xi_k) + c_{11}(\rho)(\delta_{ik}\delta_{jl}\delta_{pr}\xi_s\xi_t + \\
& + \delta_{ik}\delta_{jl}\delta_{st}\xi_p\xi_r + \delta_{il}\delta_{jk}\delta_{pr}\xi_s\xi_t + \delta_{il}\delta_{jk}\delta_{st}\xi_p\xi_r + \delta_{ps}\delta_{rt}\delta_{ij}\xi_k\xi_l + \\
& + \delta_{ps}\delta_{rt}\delta_{kl}\xi_i\xi_j + \delta_{pt}\delta_{rs}\delta_{ij}\xi_k\xi_l + \delta_{pt}\delta_{rs}\delta_{kl}\xi_i\xi_j) + c_{12}(\rho)(\delta_{ik}\delta_{jl}\delta_{ps}\xi_r\xi_t + \\
& + \delta_{ik}\delta_{jl}\delta_{pt}\xi_r\xi_s + \delta_{ik}\delta_{jl}\delta_{rs}\xi_p\xi_t + \delta_{ik}\delta_{jl}\delta_{rt}\xi_p\xi_s + \delta_{il}\delta_{jk}\delta_{ps}\xi_r\xi_t + \\
& + \delta_{il}\delta_{jk}\delta_{pt}\xi_r\xi_s + \delta_{il}\delta_{jk}\delta_{rs}\xi_p\xi_t + \delta_{il}\delta_{jk}\delta_{rt}\xi_p\xi_s + \delta_{ps}\delta_{rt}\delta_{ik}\xi_j\xi_l + \\
& + \delta_{ps}\delta_{rt}\delta_{il}\xi_j\xi_k + \delta_{ps}\delta_{rt}\delta_{jk}\xi_i\xi_l + \delta_{ps}\delta_{rt}\delta_{jl}\xi_i\xi_k + \delta_{pt}\delta_{rs}\delta_{ik}\xi_j\xi_l + \\
& + \delta_{pt}\delta_{rs}\delta_{il}\xi_j\xi_k + \delta_{pt}\delta_{rs}\delta_{jk}\xi_i\xi_l + \delta_{pt}\delta_{rs}\delta_{jl}\xi_i\xi_k) + c_{13}(\rho)\,\delta_{ij}\delta_{kl}\delta_{pr}\delta_{st} + \\
& + c_{14}(\rho)(\delta_{ij}\delta_{kl}\delta_{ps}\delta_{rt} + \delta_{ij}\delta_{kl}\delta_{pt}\delta_{rs} + \delta_{ik}\delta_{jl}\delta_{pr}\delta_{st} + \delta_{il}\delta_{jk}\delta_{pr}\delta_{st}) + \\
& + c_{15}(\rho)(\delta_{ik}\delta_{jl}\delta_{ps}\delta_{rt} + \delta_{ik}\delta_{jl}\delta_{pt}\delta_{rs} + \delta_{il}\delta_{jk}\delta_{ps}\delta_{rt} + \delta_{il}\delta_{jk}\delta_{pt}\delta_{rs})
\end{aligned}
$$

$$(\rho^2 = \xi_j\xi_j)$$

In the case of small nonhomogeneity, when expansion (3.4) may be limited to a single term and when condition (3.10) is fulfilled, the coefficents in relations (3.6) and (3.8) connecting the statistical characteristics of the state of stress and strain as well as those in (2.7) and (3.7), are determined only by the Green tensor and by the first two moments $\langle c_{ijkl}\rangle$ and c_{ijkl}^{prst} of the tensor field of the elastic moduli; relations (4.1) and (4.2) permit their calculation.

We also consider the case where the correlation tensor (1.4) does not depend on the orientation of the vector ξ, and is only a function of its modulus $\rho = \sqrt{\xi_j\xi_j}$. The above is the case related to a strong isotropy of the field, c_{ijkl}. It has been considered [4] in connection with an investigation of the elastic modulus of a polycrystalline body. For a strongly isotropic field c_{ijkl}, taking account of the symmetry in (1.3) and (1.5), we find

$$c_{ijkl}^{prst}(\rho) = h_1(\rho)\,\delta_{ij}\delta_{kl}\delta_{pr}\delta_{st} +$$
$$+h_2(\rho)\,(\delta_{ij}\delta_{kl}\delta_{ps}\delta_{rt} + \delta_{ij}\delta_{kl}\delta_{pt}\delta_{rs} + \delta_{ik}\delta_{jl}\delta_{pr}\delta_{st} + \delta_{il}\delta_{jk}\delta_{pr}\delta_{st}) +$$
$$+ h_3(\rho)\,(\delta_{ik}\delta_{jl}\delta_{ps}\delta_{rt} + \delta_{ik}\delta_{jl}\delta_{pt}\delta_{rs} + \delta_{il}\delta_{jk}\delta_{ps}\delta_{rt} + \delta_{il}\delta_{jk}\delta_{pt}\delta_{rs})$$

The calculation of the coefficients entering into relations (3.6) and (3.8) is appreciably simplified in the case of strong isotropy. In particular, if the expansion (3.4) is limited to one term, calculation for $\langle\psi_{ijst}\rangle$ in (3.6) gives

$$\langle\psi_{ijst}\rangle = -\,{}^4/_3\,\pi\,(\lambda_1 + {}^1/_5\lambda_2)\,c_{nkst}^{ijnk}(0) + {}^8/_{15}\,\pi\lambda_2 c_{nnst}^{ijmm}(0) \tag{4.3}$$

where λ_1 and λ_2 are expressed in terms of the constants μ_1 and μ_2 in (4.1) by Formulas

$$\lambda_1 = \frac{\mu_1 + 3\mu_2}{8\pi\mu_2(\mu_1 + 2\mu_2)}, \qquad \lambda_2 = \frac{\mu_1 + \mu_2}{8\pi\mu_2(\mu_1 + 2\mu_2)}$$

Relation (4.3) coincides with the result obtained in a different way in [4].

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